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APPROXIMATE FORMULAS FOR THE CALCULATION OF FUNCTIONALS OF GAUSSIAN RANDOM PROCESSES AND NUMERICAL EXAMPLES

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INTRODUCTION

One of the problems of approximate stochastic analysis is the construction of the numerical methods for the calculation of probabilistic characteristics (mathematical expectation, dispersion, etc) of random processes. In various applications the processes of the form $\eta(t) = F\Big(\xi(t)\Big)$, where $\xi(t)$ is some fixed random process with a given average and dispersion, $F(x)$ is the function defined on $\mathbb{R}$, are often used. Such processes arise, in particular, and when solving stochastic differential equations.

In our message, we will consider random processes $\eta(t)$, where $\xi(t)$ are some processes generated by the Brownian motion process $W(t)$ (Wiener process). Further we give the methods of approximate calculation of averages for such processes and numerous examples.

$\xi(t)$ – THE BROWNIAN BRIDGE $B(t)$

The random process $B(t) = W(t) - tW(1)$, where $W(t)$, $0 \le t \le 1$ is standard Wiener process, is related to the set of processes called the Brownian bridge.

Its probabilistic characteristics:

$$E\Big\{B(t)\Big\} = 0, \;\; D\Big[B(t)\Big] = t(1-t),$$

$$Cov\Big[B(s),\, B(t)\Big] = s(1-t), \;\; s \le t.$$

The random process of the form $\xi(t) = F\Big(B(t)\Big)$, where $y = F(x)$ is the function, given on $\mathbb{R}$, as well as some other processes $\xi(t)$ of such structure are stochastic models of many real phenomena and processes.

THE BROWNIAN BRIDGE OF MORE GENERAL FORM

Let there be set three real numbers: α, β and $T > 0$. The continuous Gaussian process $\xi(t)$ $\Big(0 \le t \le T,\ \xi(0) = \alpha\Big)$, for which

$$E\Big\{\xi(t)\Big\} = \alpha + (\beta - \alpha)\frac{t}{T},\ \ Cov\Big[\xi(t),\, \xi(s)\Big] = \min(s,t) - \frac{st}{T},$$

is called *the Brownian bridge* (*fixed Brownian motion*) from α to β of length T.

It is obvious that $E\Big\{\xi(t)\Big\} = \beta$, and $Cov\Big[\xi(t),\, \xi(s)\Big] = 0$, for $s = T$ or $t = T$. Hence, it is almost surely that $\xi(t) = \beta$, that is the main property of random processes of this type. The random process $\xi(t)$ with $\alpha = \beta = 0$ and $T = 1$ coincides with $B(t)$.

EXAMPLES OF THE BROWNIAN BRIDGE

The processes defined by the following two formulas

$$\xi_1(t) = \alpha + W(t) - \frac{t}{T}W(T),$$

$$\xi_2(t) = \alpha + \frac{T-t}{T}W\left(\frac{T}{T-t}\right), \;\; 0 \le t \le T,$$

$$\xi_3(t) = (1-t)W\left(\frac{t}{1-t}\right) \;\; (0 \le t \le 1),$$

also represent the Brownian bridge.

We consider Brownian bridge $B(t)$ $(0 \leq t \leq 1)$ as a Gaussian process with zero average and dispersion $t(1-t)$. Consequently,

$$E\Big\{F\Big(B(t)\Big)\Big\} = \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} e^{-x^2} F\left(\sqrt{2t(1-t)}x\right) dx, \tag{1}$$

where $F(x)$ is the function given on $\mathbb{R}$, for which this integral exists.

With $F(x) = x^{2k+1}$ $(k = 0, 1, 2, \ldots)$ the integral (1) vanishes. For $F(x) = x^{2k}$ we have:

$$E\Big\{B^{2k}(t)\Big\} = \frac{2^k t^k (1-t)^k}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} e^{-x^2} x^{2k} dx =$$

$$= (2k-1)!! t^k (1-t)^k \ \ (k = 0, 1, \ldots). \tag{2}$$

From (1), in particular, we also get $E\left\{e^{\lambda B(t)}\right\} = e^{\frac{1}{2}\lambda^2 t(1-t)}$.

Brownian bridge $\xi(t)$ from α to β can be also defined as the solution of stochastic differential equations. One of the simplest variants is equation

$$d\xi(t) = \frac{\beta - \xi(t)}{T - t}dt + dW(t), 0 \le t \le T, \ \ \xi(0) = \alpha, \tag{3}$$

the solution $\xi(t)$ of which has the form

$$\xi(t) = \alpha + (\beta - \alpha)\frac{t}{T} + (T - t)\int_0^t \frac{dW(s)}{T - s}.$$

Here

$$\xi(T) = \beta,$$

$$E\Big\{\xi(t)\Big\} = \alpha + (\beta - \alpha)\frac{t}{T},$$

$$D(\xi) = \frac{t}{T}(T - t).$$

ON SOME FORMULAS OF APPROXIMATING CONSIDERED RANDOM PROCESSES

Let $\xi = \xi(t)$ $(0 \le t \le T, T > 0)$ be a real Gaussian random process, $E\Big\{\xi(t)\Big\} = m = m(t)$ and $D(\xi) = \sigma = \sigma(t)$.
Then for process $\eta(t) = F\Big(\xi(t)\Big)$

$$\tilde{m}(t) = E\Big\{\eta(t)\Big\} = I_1(t), \quad \tilde{\sigma}(t) = D(\eta) = I_2(t) - \tilde{m}^2(t), \qquad (4)$$

where

$$I_1(t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-s^2} F\left(\sqrt{2\sigma}s + m\right) ds,$$

$$I_2(t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-s^2} F^2\left(\sqrt{2\sigma}s + m\right) ds,$$

and function $F(x)$ on $\mathbb{R}$ is that integrals $I_1(t)$ and $I_2(t)$ exist.

For approximate calculation of integrals $I_1(t)$ and $I_2(t)$ one usually uses quadrature formulas. For example, for $I_1(t)$:

$$I_1(t) = \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} e^{-x^2} F\left(\sqrt{2\sigma}x + m\right) dx =$$

$$= \frac{1}{\sqrt{\pi}} \sum_{k=1}^{n} A_k F\left(\sqrt{2\sigma}x_k + m\right) + r_n(F), \tag{5}$$

where x_k are the roots of Chebyshev–Hermite polynomial $H_n(x) = = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$, $A_k = \sqrt{\pi} 2^{n+1} n! \left[H_n'(x_k)\right]^{-2}$ $(k = 1, 2, \ldots, n)$.

For the error $r_n(F)$ equality holds

$$r_n(F) = \frac{n!\sqrt{\pi}}{2^n (2n)!} F^{(2n)}(\eta),$$

$$F = F\left(\sqrt{2\sigma}x + m\right), \quad \eta \in \mathbb{R}.$$

We introduce the following notations

$$S_n(t) = \frac{1}{\sqrt{\pi}} \sum_{k=1}^{n} A_k F\left(\sqrt{2\sigma} x_k + m\right),$$

$$S_n^{(2)}(t) = \frac{1}{\sqrt{\pi}} \sum_{k=1}^{n} A_k F^2\left(\sqrt{2\sigma} x_k + m\right).$$

Let us consider the following random process sequences:

$$\eta_n(t) = \sum_{k=1}^{n} l_{nk}\left(\frac{W(t)}{\sqrt{2t}}\right) F\left(\sqrt{2\sigma} x_k + m\right),$$

$$\eta_n^{(2)}(t) = \sum_{k=1}^{n} l_{nk}\left(\frac{W(t)}{\sqrt{2t}}\right) F^2\left(\sqrt{2\sigma} x_k + m\right),$$

$$\tilde{\eta}_n(t) = \sum_{k=1}^{n} l_{nk}\left(\frac{B(t)}{\sqrt{2t(1-t)}}\right) F\left(\sqrt{2\sigma} x_k + m\right),$$

$$\tilde{\eta}_n^{(2)}(t) = \sum_{k=1}^{n} l_{nk} \left(\frac{B(t)}{\sqrt{2t(1-t)}} \right) F^2 \left(\sqrt{2\sigma} x_k + m \right).$$

Here $l_{nk}(x) = \frac{(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$.

Theorem. *Let function* $F\left(\sqrt{2\sigma} x_k + m\right)$ *be a continuous one on the real axis and integrals in*

$$\tilde{m}(t) = E\Big\{\eta(t)\Big\} = I_1(t), \quad \tilde{\sigma}(t) = D(\eta) = I_2(t) - \tilde{m}^2(t),$$

exist. Then the sequences of averages $E\Big\{\eta_n(t)\Big\}$, $E\Big\{\tilde{\eta}_n(t)\Big\}$ *at* $n \to \infty$ *converge on* T *to average* $E\Big\{\eta(t)\Big\}$*, and sequences* $E\left\{\eta_n^{(2)}(t)\right\}$, $E\left\{\tilde{\eta}_n^{(2)}(t)\right\}$ *correspondingly converge to second moment* $E\left\{\eta^2(t)\right\}$ *of process* $\eta(t) = F\Big(\xi(t)\Big)$*, where* $\xi(t)$ *is a Gaussian random process with average* $m = m(t)$ *and dispersion* $\sigma = \sigma(t)$, $(0 \le t \le 1)$.

NUMERICAL EXAMPLES

Let $\eta(t) = \sin\Big(\xi(t)\Big)$, where $\xi(t)$ is an Ornstein–Uhlenbeck process, defined as the solution of stochastic differential equation

$$d\xi(t) = -\alpha\Big(\xi(t) - \beta\Big)dt + \gamma dW(t) \ \ (t \leq 0)$$

with initial condition $\xi(0) = \xi_0$, where α, β, γ and ξ_0 are fixed numerical values. Solution $\xi(t)$ is defined by formula

$$\xi(t) = \xi_0 e^{-\alpha t} + \beta\Big(1 - e^{-\alpha t}\Big) + \gamma \int_0^t e^{-\alpha(t-s)} dW(s),$$

and average $m = m(t)$ and dispersion $\sigma = \sigma(t)$ correspondingly have the form of

$$m = \xi_0 e^{-\alpha t} + \beta\Big(1 - e^{-\alpha t}\Big),$$

$$\sigma = \frac{\gamma^2}{2\alpha}\left(1 - e^{-2\alpha t}\right). \tag{6}$$

EXAMPLE 1

Let $\xi(t)$ be an Ornstein–Uhlenbeck process. For random process $\eta(t) = \sin\Big(\xi(t)\Big)$ let us calculate mathematical expectation

$$E\Big\{\eta(t)\Big\} = \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} e^{-x^2} \sin\left(\sqrt{2\sigma}x + m\right) dx \qquad (7)$$

and its approximate value

$$S_n(t) = \frac{1}{\sqrt{\pi}} \sum_{k=1}^{n} A_k \sin\left(\sqrt{2\sigma}x_k + m\right). \qquad (8)$$

Integral (7) is calculated exactly: $E\Big\{\eta(t)\Big\} = e^{-\frac{1}{2}\sigma} \sin m$, where

$$m = \xi_0 e^{-\alpha t} + \beta\Big(1 - e^{-\alpha t}\Big), \quad \sigma = \frac{\gamma^2}{2\alpha}\left(1 - e^{-2\alpha t}\right).$$

We present the numerical values of error

$$r_n(t) = e^{-\frac{1}{2}\sigma(t)} \sin m(t) - S_n(t)$$

at points $t_i = \dfrac{i}{10}$ $(i = 0, 1, \ldots, 10)$ for $n = 10$. Here

$$S_n(t) = \frac{1}{\sqrt{\pi}} \sum_{k=1}^{n} A_k \sin\left(\sqrt{2\sigma} x_k + m\right).$$

Calculation results at $\xi_0 = 1, \alpha = 2, \beta = 3, \gamma = 5$ are given in Table 1.

t_i	$E\{\eta(t_i)\}$	$S_n(t_i)$	$r_n(t_i)$
0	0.84147098481	0.841470985	-1.9×10^{-10}
0.1	0.34920570709	0.349205706	1.1×10^{-9}
0.2	0.17821334508	0.178213194	1.5×10^{-7}
0.3	0.10648205980	0.106480813	1.2×10^{-6}
0.4	0.07122171563	0.071218014	3.7×10^{-6}
0.5	0.05157706331	0.051570409	6.7×10^{-6}
0.6	0.03950761433	0.039498551	9.1×10^{-6}
0.7	0.03150780929	0.031497321	1.0×10^{-5}
0.8	0.02588852563	0.025877529	1.1×10^{-5}
0.9	0.02176554463	0.021754685	1.1×10^{-5}
1.0	0.01864185530	0.018631506	1.0×10^{-5}

EXAMPLE 2

For process $\eta(t) = \cos^2\Big\{\xi(t)\Big\}$ the average is defined by formula

$$E\Big\{\eta(t)\Big\} = \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} e^{-s^2} \cos^2\Big(\sqrt{2\sigma} s + m\Big)\, ds = \frac{1}{2}\Big(1 + e^{-2\sigma} \cos 2m\Big),$$

and the approximate value in this case is

$$S_n(t) = \frac{1}{\sqrt{\pi}} \sum_{k=1}^{n} A_k \cos^2\Big(\sqrt{2\sigma} x_k + m\Big). \qquad (9)$$

Here error

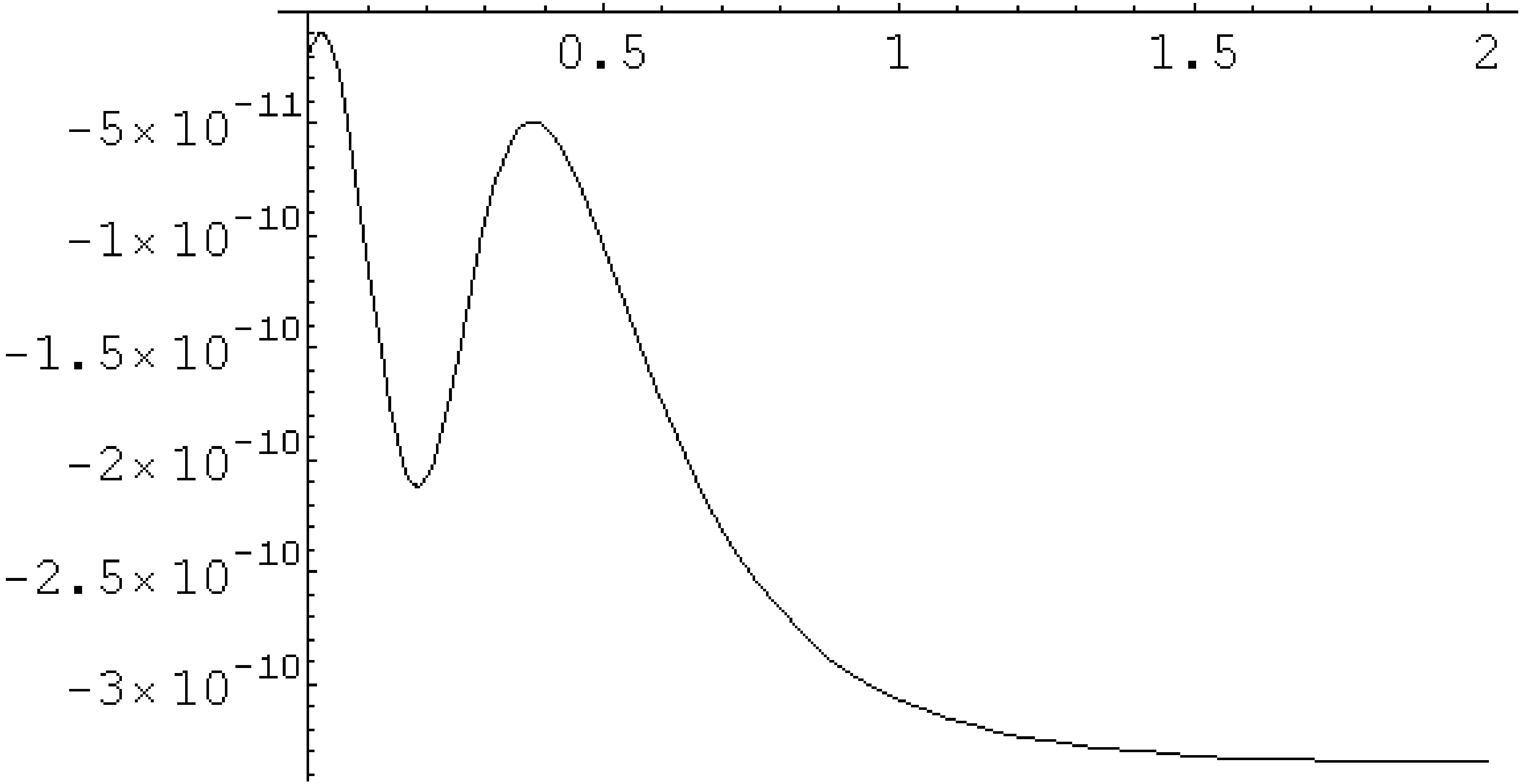
$$r_n(t) = \frac{1}{2}\Big(1 + e^{-2\sigma(t)} \cos 2m(t)\Big) - S_n(t),$$

points $t_i = \dfrac{i}{5}$ $(i = 0, 1, \ldots, 10)$.

Results for parameter values equal to $\xi_0 = 5$, $\alpha = 4$, $\beta = 2$, $\gamma = 2$, at given points t_i of segment $[0; 2]$.

t_i	$E\{\eta(t_i)\}$	$S_n(t_i)$	$r_n(t_i)$
0	0.08046423546	0.080464235	4.6×10^{-10}
0.2	0.70618461865	0.706184619	-3.5×10^{-10}
0.4	0.59168445417	0.591684454	1.7×10^{-10}
0.6	0.46897426165	0.468974262	-3.6×10^{-10}
0.8	0.41691669322	0.416916693	2.2×10^{-10}
1.0	0.39572637760	0.395726378	-4.0×10^{-10}
1.2	0.38677884287	0.386778843	-1.3×10^{-10}
1.4	0.38288530187	0.382885302	-1.3×10^{-10}
1.6	0.38116239149	0.381162392	-5.1×10^{-10}
1.8	0.38039368806	0.380393688	6.3×10^{-11}
2.0	0.38004939542	0.380049396	-5.9×10^{-10}

Here is the graph of error $r_n(t)$



OTHER METHODS OF APPROXIMATION

Let's consider another class of random processes

$$\tilde{\eta}_n^{(1)}(t) =$$

$$= \frac{1}{\sqrt{\pi}} \sum_{k=0}^{n} C_n^k \frac{(1-t)^{n-2k}}{(2k-1)!!} B^{2k}(t) \int_{-\infty}^{\infty} e^{-x^2} F\left(\sqrt{2\sigma\left(\frac{k}{n}\right)}x + m\left(\frac{k}{n}\right)\right) dx, \tag{10}$$

where $B(t) = W(t) - tW(1)$, $C_n^k = \dfrac{n!}{k!(n-k)!}$, $(k = 1, 2, \ldots, n;\ n = 1, 2, \ldots)$, $0 \le t \le 1$. Mathematical expectation $\tilde{m}_n^{(1)}(t) = E\left\{\tilde{\eta}_n^{(1)}(t)\right\}$ of process (10) is determined by the formula

$$\tilde{m}_n^{(1)}(t) = \sum_{k=0}^{n} C_n^k t^k (1-t)^{n-k} \tilde{m}\left(\frac{k}{n}\right). \tag{11}$$

Hence, $\lim\limits_{n\to\infty} \tilde{m}_n^{(1)}(t) = E\left\{\eta(t)\right\} = \tilde{m}(t)$.

In the function approximation theory of E.V. Voronovskaya's formula concerning the asymptotic specification of function approximations by Bernstein polynomials:

$$B_n(f;t) = f(t) + \frac{1}{2}\frac{t-t^2}{n} f''(t) + \frac{\varepsilon_n}{n},$$

where $\varepsilon_n \to 0$ at $n \to \infty$ is known.

We consider approximate formula

$$\widehat{B}_n(f;t) = B_n(f;t) - \frac{1}{2}\frac{t-t^2}{n} f''(t) \approx f(t) \quad (0 \le t \le 1)$$

and corresponding random processes

$$\widehat{\eta}_n(t) = \tilde{\eta}_n^{(1)}(t) - \frac{1}{2n\sqrt{\pi}} B^2(t) \int\limits_{-\infty}^{\infty} e^{-x^2} F''\left(\sqrt{2\sigma(t)}x + m(t)\right) dx$$

$$(n = 1, 2, \ldots). \tag{12}$$

Mathematical expectation $\widehat{m}_n(t)$ of processes

$$\widehat{\eta}_n(t) = \tilde{\eta}_n^{(1)}(t) - \frac{1}{2n\sqrt{\pi}} B^2(t) \int\limits_{-\infty}^{\infty} e^{-x^2} F'' \left(\sqrt{2\sigma(t)}x + m(t) \right) dx$$

$$(n = 1, 2, \ldots) \qquad (12)$$

will be defined by formula

$$\widehat{m}_n(t) = B_n \left(\tilde{m}_n^{(1)}(t); t \right) -$$

$$- \frac{1}{2\sqrt{\pi}} \frac{t - t^2}{n} \int\limits_{-\infty}^{\infty} e^{-x^2} F'' \left(\sqrt{2\sigma(t)}x + m(t) \right) dx. \qquad (13)$$

Consequently, $\lim\limits_{n\to\infty} E\Big\{\widehat{\eta}_n(t)\Big\} = E\Big\{\eta(t)\Big\}$.

EXAMPLE 3

Let $F(x) = \sin^2 x$, i. e., $\eta(t) = \sin^2 \xi(t)$, where $\xi(t)$ is an Ornstein–Uhlenbeck process.

$$\tilde{m}_s(t) = E\Big\{\eta(t)\Big\} = \frac{1}{2}\Big(1 - e^{-2\sigma(t)} \cos 2m(t)\Big),$$

and

$$m = \xi_0 e^{-\alpha t} + \beta\Big(1 - e^{-\alpha t}\Big), \;\; \sigma = \frac{\gamma^2}{2\alpha}\Big(1 - e^{-2\alpha t}\Big),$$

where $\xi_0 = 4$, $\alpha = 5$, $\beta = 2$, $\gamma = 1$.

The calculation results by formula

$$\tilde{m}_n^{(1)}(t) = \sum_{k=0}^{n} C_n^k t^k (1-t)^{n-k} \tilde{m}\left(\frac{k}{n}\right), \tag{11}$$

where $\tilde{m}(t) = \tilde{m}_s(t)$, at points $t_i = \dfrac{i}{5}$ $(i = 0, 1, \ldots, 5)$ of segment $[0, 1]$ at $n = 20$ are given in

Table 3

t_i	$E\{\eta(t_i)\}$	$\tilde{m}_n^{(1)}(t_i)$	$r_n(t_i)$
0	0.5728	0.5728	0.0000
0.2	0.2105	0.2435	−0.0330
0.4	0.5699	0.5364	0.0336
0.6	0.7011	0.6878	0.0133
0.8	0.7442	0.7413	0.0029
1.0	0.7591	0.7591	0.0000

and in accordance with a more precise formula

$$\widehat{m}_n(t) = B_n\left(\tilde{m}_n^{(1)}(t); t\right) -$$

$$-\frac{1}{2\sqrt{\pi}}\frac{t-t^2}{n}\int\limits_{-\infty}^{\infty} e^{-x^2} F''\left(\sqrt{2\sigma(t)}x + m(t)\right) dx,$$

in Table 4

t_i	$E\{\eta(t_i)\}$	$\widehat{m}_n(t_i)$	$\tilde{r}_n(t_i)$
0	0.5728	0.5728	0.0000
0.2	0.2105	0.2118	–0.0013
0.4	0.5699	0.5732	–0.0032
0.6	0.7011	0.7048	–0.0037
0.8	0.7442	0.7467	–0.0025
1.0	0.7591	0.7591	0.0000

Here $r_n(t_i) = E\{\eta(t_i)\} - \tilde{m}_n^{(1)}(t_i)$, while $\tilde{r}_n(t_i) = E\{\eta(t_i)\} - \widehat{m}_n(t_i)$.

In this case there has been used the exact value of the integral in (12) and (13), and the calculations were made by formula

$$\widehat{m}_n(t) = \sum_{k=0}^{n} C_n^k t^k (1-t)^{n-k} \tilde{m}_s(t) + \frac{1}{2n}\left(t - t^2\right) m_{2s}(t),$$

where $m_{2s}(t) = \dfrac{1}{2}\left(1 - e^{-8\sigma(t)} \cos 2m(t)\right).$

THE CALCULATION OF THE MOMENTS OF THE STOCHASTIC PROCESSES DEFINED BY TRIGONOMETRIC FUNCTIONS OF THE BROWNIAN MOTION

Analogous problems for the processes of the form

$$\eta(t) = F\Big(\alpha W^2(t) + \beta W(t) + \gamma\Big), \tag{14}$$

where $F(x)$ is the function given and continuous on $\mathbb{R}$, $W(t)$ is also standard Wiener process, are considered.

QUADRATURE FORMULAS FOR THE CALCULATION OF THE MATHEMATICAL EXPECTATION

We use the well-known formula for the calculation of k-th moments of the random process (14):

$$m_k(t) = E\Big\{\xi^k(t)\Big\} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} F^k\Big(2\alpha t x^2 + \beta\sqrt{2t}x + \gamma\Big)dx, \quad (15)$$

($k = 0, 1, 2, \ldots$), the function $F(x)$ and parameters α, β, γ are such that the integral (15) converges.

For the approximate calculation of the integral (15) we use the quadrature formula of the highest algebraic degree of accuracy for the integrals over the number axis with the weighting function $p(x) = e^{-x^2}$.

The approximate calculation of $m_k(t)$ can be made from the formula

$$m_k(t) = \frac{1}{\sqrt{\pi}} \sum_{\nu=1}^{n} A_\nu F^k\Big(2\alpha t x_\nu^2 + \beta\sqrt{2t}x_\nu + \gamma\Big) + \frac{1}{\sqrt{\pi}} r_n(F^k), \qquad (16)$$

where x_ν are the roots of the Hermite polynomial of n-th power $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$, A_ν are the coefficients of the quadrature formula, and $r_n(F^k)$ its remainder term.

We denote the quadrature sum in the formula (16) by

$$S_{nk}(t) = \frac{1}{\sqrt{\pi}} \sum_{\nu=1}^{n} A_\nu F^k\Big(2\alpha t x_\nu^2 + \beta\sqrt{2t}x_\nu + \gamma\Big). \qquad (17)$$

We consider the sequence of the random processes

$$\eta_{nk}(t) = \sum_{\nu=1}^{n} l_{n\nu}\left(\frac{W(t)}{\sqrt{2t}}\right) F^k\Big(2t\alpha x_\nu^2 + \sqrt{2t}\beta x_\nu + \gamma\Big) \quad (n, k = 1, 2, \ldots), \tag{18}$$

where $l_{n\nu}(x) = \frac{(x-x_1)\cdots(x-x_{\nu-1})(x-x_{\nu+1})\cdots(x-x_n)}{(x_\nu-x_1)\cdots(x_\nu-x_{\nu-1})(x_\nu-x_{\nu+1})\cdots(x_\nu-x_n)}$.

For the calculation of the mathematical expectation of the random process $\eta(t)$ of the form $\eta(t) = f\Big(W(t)\Big)$ we use the formula

$$E\Big\{\eta(t)\Big\} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-x^2} f\Big(\sqrt{2t}x\Big)dx, \tag{19}$$

where $y = f(x)$ is the function for which the improper integral on the right-hand side of (19) converges.

Let us consider examples of random processes and give the results of the computational experiment.

For the random process

$$\eta(t) = \theta \cos\Big(\alpha W^2(t) + \beta W(t) + \gamma\Big), \tag{20}$$

where θ is arbitrary fixed constant, using (15) and the formula for the calculation of this kind of integrals we can calculate the first order moment (the mathematical expectation):

$$m_c(t) = E\Big\{\eta(t)\Big\} = \frac{\theta}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} \cos\Big(2\alpha t x^2 + \beta\sqrt{2t}x + \gamma\Big) dx =$$

$$= \frac{\theta}{\sqrt[4]{1+4\alpha^2 t^2}} \exp\left\{-\frac{\beta^2 t}{2(1+4\alpha^2 t^2)}\right\} \times$$

$$\times \cos\left\{\frac{1}{2} arctg(2\alpha t) + \frac{(4\alpha^2\gamma - \alpha\beta^2)t^2 + \gamma}{1+4\alpha^2 t^2}\right\}, \ t \in [0;1].$$

And, correspondingly, the approximate process $\eta_{nc}(t)$ for (20), by virtue of the formula (18) has the following form

$$\eta_{nc}(t) = \theta \sum_{\nu=1}^{n} l_{n\nu} \left(\frac{W(t)}{\sqrt{2t}} \right) \cos\Bigl(2t\alpha x_\nu^2 + \beta\sqrt{2t} x_\nu + \gamma \Bigr),$$

and the quadrature sum (17) $S_{nc}(t)$ will be given, respectively, by the formula

$$S_{nc}(t) = \frac{\theta}{\sqrt{\pi}} \sum_{\nu=1}^{n} A_\nu \cos\Bigl(2\alpha t x_\nu^2 + \beta\sqrt{2t} x_\nu + \gamma \Bigr),$$

which is an approximation to the exact mathematical value $m_c(t)$.

EXAMPLE 4

For the random process (20), the value of the error $r_{nc}(t) = m_c(t) - S_{nc}(t)$ of the calculation of the mathematical expectation $m_c(t)$ at the points $t_i = \frac{i}{5}$ $(i = 0, 1, \ldots, 5)$ for $\alpha = \beta = \gamma = \theta = 1$ and $n = 9$ is given in

Table 5

t_i	$S_{nc}(t_i)$	$m_c(t_i)$	$r_{nc}(t_i)$
0	0,54043931	0,54030231	–0,00013701
0,2	0,35652498	0,356435971	$-8{,}90121 \times 10^{-5}$
0,4	0,25443564	0,254201549	–0,00023409
0,6	0,20050560	0,195744407	–0,0047612
0,8	0,1536785	0,1574841	0,0038057
1	0,07416710	0,13041004	0,05624294

For the random process

$$\zeta(t) = \theta \sin\Big(\alpha W^2(t) + \beta W(t) + \gamma\Big), \tag{21}$$

analogously we can calculate exact value of the first order moment:

$$m_s(t) = E\Big\{F(W(t))\Big\} = \frac{\theta}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} \sin\Big(2\alpha t x^2 + \beta\sqrt{2t}x + \gamma\Big) dx =$$

$$= \frac{\theta}{\sqrt[4]{1 + 4\alpha^2 t^2}} \exp\left\{ -\frac{\beta^2 t}{2(1 + 4\alpha^2 t^2)} \right\} \times$$

$$\times \sin\left\{ \frac{1}{2} arctg(2\alpha t) + \frac{(4\alpha^2\gamma - \alpha\beta^2)t^2 + \gamma}{1 + 4\alpha^2 t^2} \right\}.$$

And, correspondingly, the approximation $\eta_{ns}(t)$ for the process (21) will be

$$\eta_{ns}(t) = \theta \sum_{\nu=1}^{n} l_{n\nu} \left(\frac{W(t)}{\sqrt{2t}} \right) \sin\Big(2\alpha t x_\nu^2 + \beta \sqrt{2t} x_\nu + \gamma \Big),$$

and $S_{ns}(t)$ will be given by the formula

$$S_{ns}(t) = \frac{\theta}{\sqrt{\pi}} \sum_{\nu=1}^{n} A_\nu \sin\Big(2\alpha t x_\nu^2 + \beta \sqrt{2t} x_\nu + \gamma \Big).$$

THE BIBLIOGRAPHY

1. T. Hida. *Brownian motion.* (Nauka Publ. Chief ed. board of physical and mathematical lit-re, Moscow, 1987).

2. A.N. Borodin, P. Salminen. *Reference book on Brownian motion. Facts and formulas.* (Lan', St. Petersburg, 2000).

3. L.A. Yanovich. *Approximate calculation of continual integrals on Gauss measures.* (Science and technology, Minsk, 1976).

4. I.S. Gradshtein, I.M. Ryzhik. *Tables of integrals, sums, series and products.* (PhysMathGiz, Moscow, 1962).

5. Xuerong Mao. *Stochastic Differential Equations and Applications. Second Edition.* (Woodhead Publishing, 2008).

THE BIBLIOGRAPHY

6. V.I. Krylov. *Approximate calculation of integrals.* (Nauka Publ. Chief ed. board of physical and mathematical lit-re, Moscow, 1967).

7. V.I. Krylov, L.T. Shulgina. *Handbook on numerical integration.* (Nauka Publ. Chief ed. board of physical and mathematical lit-re, 1966).

8. L.A. Yanovich, I.N. Gulo. *On the approximate calculation of functions of the Brownian motion process.* (Vesci NAN Belarusi, 2014).

9. P.P. Korovkin. *Linear operators and approximation theory.* (Delhi: Hindustan Publ. Corp., 1960).

10. F. Altomare, M. Campiti. *Korovkin-type Approximation Theory and its Applications.* (Berlin: New York: de Gruyter, 1994).

THANK YOU FOR ATTENTION