

## MODELLING OF QUASI-CHERENKOV ELECTRON BEAM INSTABILITY IN PERIODICAL STRUCTURES<sup>1</sup>

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**Abstract.** Nonlinear stage of quasi-Cherenkov instability of electron beam under conditions of two- and three-dimensional distributed feedback is simulated. The scheme of distributed feedback with two strong coupled waves is considered. Mathematical model of quasi-Cherenkov electron beam instability is proposed. Numerical method to solve the nonlinear integro-differential system, describing such instability, is worked out. Results of numerical experiments are discussed.

**Key words:** quasi-Cherenkov instability, numerical modelling, nonlinear integro-differential system

### 1. Introduction

This contribution is devoted to modelling of nonlinear stage of quasi-Cherenkov electron beam instability under the conditions of two- and three-dimensional distributed feedback. Quasi-Cherenkov instability takes place when one or more wave refraction index satisfy the Cherenkov condition [4]. In this case electrons radiate coherently. Such instability mechanism can be considered as a technique for realization of Free Electron Laser (FEL). FELs are devices which use the electron beam energy to generate coherent electromagnetic radiation. Such devices are very perspective for electromagnetic radiation generation in wide spectral range. Nowadays FEL lasing is obtained in different wavelength ranges: from centimeter to ultra-violet [2, 9, 14]. The high expensive International X-ray FEL project is on the preparation stage now [16]. Volume FEL (VFEL) based on the mechanism of multi wave volume distributed feedback (VDVB) was proposed in [3, 5]. VFELs give possibility to reduce the starting currents of generation, to provide generation in large volume, to tune

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generation frequency [3]. VFEL generation in large volume essentially increases the electric strength of resonator and allows to produce electromagnetic pulses of great power ( $> 10$  GWt) in *mm–cm* range. Besides the multiwave distributed feedback in VFEL provides the modes discrimination in the case when the linear sizes of resonator (waveguide) cross section exceed generated wavelength (the so-called oversized systems).

First lasing of volume FEL (VFEL) in millimeter range was recently obtained by a group of scientists of Institute for Nuclear Problems [7].

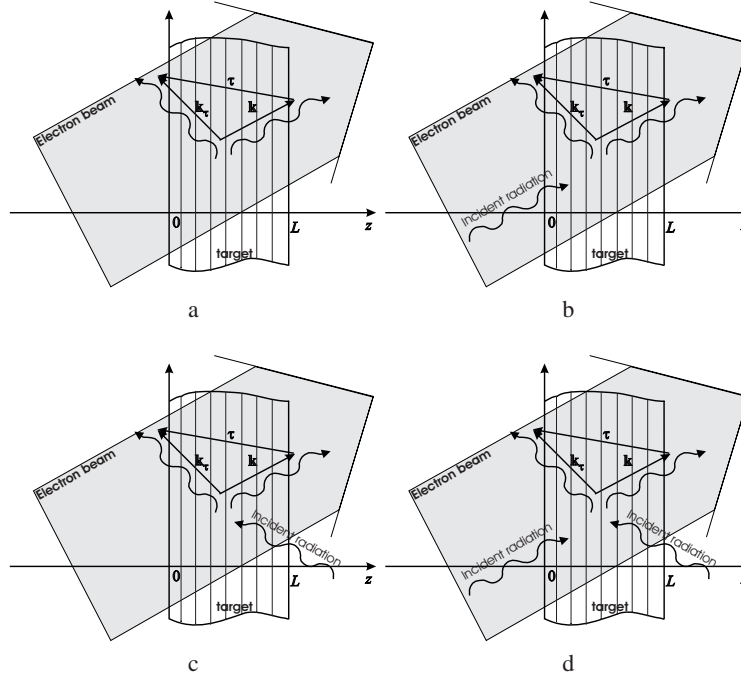
A lot of papers are devoted to FEL simulation (for example [10, 11]). In our works [1, 6, 8, 18, 19, 20, 21, 22] we have considered mathematical models of different types of VFEL in X-ray, optical and millimeter wave ranges. Earlier electron beam was simulated as a hydrodynamical approximation [1, 18] or as distribution functions [8, 19]. It turned out, however, that hydrodynamical approximation is very rough. And in millimeter range electron beam presentation as distribution functions for sufficiently large beam current density leads to appearance of non-physical instability related to the computational error. Therefore in this work simulation of quasi-Cherenkov instability is performed by means of phase averaging method which is frequently used in large number of works. The main distinction of this work is in applying such method to VFEL simulation.

## 2. Mathematical Model

Let us consider quasi-Cherenkov stimulated radiation of wide electron beam passing through spatial periodic structure. In Fig. 1 four schemes of simple VFEL are presented. A target of length  $L$  is a medium possessing spatially periodic permittivity. There are several different possibilities. Fig. 1a corresponds to the case when there are no incident waves emerging on system. For distributed feedback forming the specific so-called diffraction (or the Bragg) conditions can be fulfilled. These conditions have the form  $|\mathbf{k}| = |\mathbf{k}_\tau|$  for the two waves case. Here  $\mathbf{k}$  is the radiation wave vector and  $\tau$  is reciprocal vector of the periodical structure  $\tau = 2\pi n_1/d_1, 2\pi n_2/d_2, 2\pi n_3/d_3$ ,  $d_1, d_2, d_3$  are basic translation periods,  $n_1, n_2, n_3$  are integers. Here we consider so-called Bragg diffraction geometry when one wave propagates in forward direction and the other – in backward (see Fig. 1a – Fig. 1d).

The developed mathematical model allows to consider such geometry when incident wave emerges from the side  $z = 0$  (forward incident wave Fig. 1b), or from the side  $z = L$  (backward incident wave Fig. 1c), or from both sides simultaneously (see, Fig. 1d). Moreover two mirrors can be placed on each side of the target to accumulate radiation.

Let us consider the system of equations describing quasi-Cherenkov instability. Equations for this process are written for stationary regime of nonlinear saturation. This system with appropriate boundary conditions is written as follows:



**Figure 1.** Scheme of quasi-Cherenkov VFEL in Bragg geometry

$$\begin{cases}
 \frac{dE}{dz} + a_{11}E + a_{12}E_\tau \\
 \quad = \Phi \int_0^{2\pi} \frac{2\pi - p}{8\pi^2} (\exp(-i\Theta(z, p)) + \exp(-i\Theta(z, -p))) dp, \\
 E(0) = E_0, \quad z \in [0, L], \quad p \in [-2\pi, 2\pi], \\
 \frac{dE_\tau}{dz} + a_{21}E + a_{22}E_\tau = 0, \quad E_\tau(L) = E_1, \\
 \frac{d^2\Theta(z, p)}{dz^2} = \Psi \left( k - \frac{d\Theta(z, p)}{dz} \right)^3 \operatorname{Re}(E(z) \exp(i\Theta(z, p))), \\
 \Theta(0, p) = p, \quad \frac{d\Theta(0, p)}{dz} = 0;
 \end{cases} \quad (1)$$

where  $i$  is the imaginary unit.

There are two independent arguments in system (1): spatial coordinate  $z$  and initial electron phase  $p$ . Amplitudes of electromagnetic fields  $E(z)$ ,  $E_\tau(z)$  and coefficients  $a$  are complex-valued. Function  $\Theta(z, p)$  describes phase of electron beam relative to the electromagnetic wave.  $\Theta(z, p)$  and coefficients  $\Phi$  and  $\Psi$  are real.  $k$  is a projection of wave vector  $\mathbf{k}$  on  $z$  axis. We suppose that all functions are smooth, bounded and slowly changing.

Stationary solution under zero boundary conditions is due to the essentially non-linear contribution of electron beam. In linear regime the homogeneous system with zero boundary conditions has infinite number of solutions.

### 3. Numerical Algorithm

To solve the system of integro-differential equations with nonlinearity on right-hand sides an iterative algorithm is proposed. We use notations from [17].

Introducing in domain  $\Omega = \{0 \leq z \leq L, -2\pi \leq p \leq 2\pi\}$  uniform grids on  $z$  and  $p$ :

$$\begin{aligned}\omega_z &= \{z_i = ih_z, \quad i = 0, 1, \dots, M, \quad Mh_z = L\}, \\ \omega_p &= \{p_j = h_p j, \quad j = -N, \dots, -1, 0, 1, \dots, N, \quad h_p N = 2\pi\}.\end{aligned}$$

The discrete functions, defined on the grid, will be denoted by

$$\Theta_i^j = \Theta(z_i, p_j), \quad E_i^j = E(z_i, p_j).$$

We approximate the differential problem with the following finite-difference scheme:

$$\Theta_{zz}^j = \Psi \left( k - \Theta_z^j \right)^3 \operatorname{Re} \left( \frac{s-1}{E} \exp(i\Theta^j) \right), \quad j = 0, \pm 1, \dots, \pm N, \quad (1)$$

$$E_z^s + a_{11} E^s + a_{12} E_\tau^s = \Phi \sum_{j=0}^N c_j \left( \exp(-i\Theta^j) + \exp(-i\Theta^{-j}) \right), \quad (2)$$

$$E_{\tau z}^s + a_{21} E^s + a_{22} E_\tau^s = 0, \quad (3)$$

where  $s \geq 0$  is a number of iteration. As an initial approximation we define:

$$\Theta^0_j = h_p j, \quad E^0 = 0, \quad E_\tau^0 = 0.$$

Here  $c_j$  are coefficients of quadrature formula. We use the trapezoidal rule here.

Let us write the difference equation (1) in the following form:

$$\frac{\Theta_{i+1}^{sj} - 2\Theta_i^{sj} + \Theta_{i-1}^{sj}}{h_z^2} = \Psi \left( k - \frac{\Theta_{i+1}^{sj} - \Theta_{i-1}^{sj}}{2h_z} \right)^3 \operatorname{Re} \left( \frac{s-1}{E} \exp(i\Theta_i^j) \right), \quad (4)$$

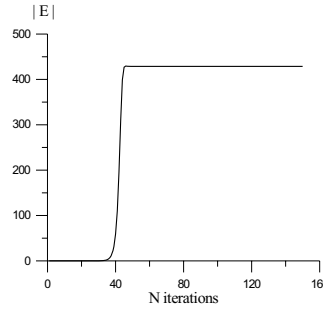
i.e. it is an implicit difference equation with respect to  $\Theta_{i+1}^{s+1}$ . Solving this cubic equation get three solutions, we choose the one which is close to  $\Theta_i^{s+1}$ . The two rest roots are meaningless. As it was shown in numerical experiments this approach works very well.

But it is not a very efficient strategy to solve numerically cubic equations. It is possible to solve it by using the Picard type iterative process:

$$\begin{cases} \frac{\Theta_{i+1}^{l+1} - 2\Theta_i^{s,j} + \Theta_{i-1}^{s,j}}{h_z^2} = \Psi \left( k - \frac{\Theta_{i+1}^l - \Theta_{i-1}^{s,j}}{2h_z} \right)^3 \operatorname{Re} \left( E^{s-1} \exp(i\Theta_i^{s,j}) \right), \\ \Theta_{i+1}^0 = \Theta_{i+1}^{s,j}, \end{cases}$$

where  $l \geq 0$  is a number of inner iterations. As it was shown in numerical experiments, it is enough to make only two inner iterations to solve the cubic equation.

Inasmuch as our iterative process (1)–(3) is nonlinear, it seems to be impossible to investigate its convergence. If we consider a linearized case of this process then some conclusions are evident. We restrict ourselves to Fig. 2, which demonstrates the convergence of the iterative process. It is stabilized after approximately 40 iterations.



**Figure 2.** Numerical solution depending on iterations number.

So, in accordance with numerical experiments, our schemes are stable and the discrete solution converges to the solution of the initial differential system because numerical results coincided in full with analytical estimations.

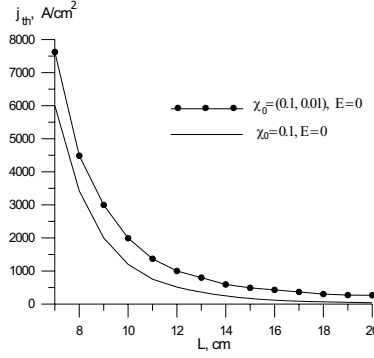
## 4. Numerical Results

Let us discuss results of numerical experiments carried out. Among them we have considered the case when there was no incident radiation, in other words  $E(0) = 0$  and  $E_\tau(L) = 0$ , as well as  $E(0) \neq 0$ .

Starting currents of electron beam, radiation power and radiation frequency depend on the feedback geometry. Therefore changing this geometry we can change these quantities and even to turn regime of generation to amplification regime and vice versa.

Threshold current density is a very important value characterizing the system. There is no generation process if current is lower than some critical value. In the case when current density is between this critical value and the value of generation threshold current ( $j_{th}$ ) the system operates in amplification regime. This is the regime of regenerative amplification for the Bragg geometry. The regime of generator is realized when the current density exceeds the threshold  $j > j_{th}$ . In that case

radiation of electron beam should exceed the losses on boundaries of the resonator and absorption losses. Such threshold is depicted in Fig.3 depending on the length of the target. It demonstrates threshold current density depending on thickness of the target with and without incident radiation and with and without absorption of the target. We can see that the larger is the target length the lower is the threshold. Absorption  $Im(\chi_0) = 0.01$  raises the threshold. Presence of incident wave with  $E \neq 0$  decreases it.

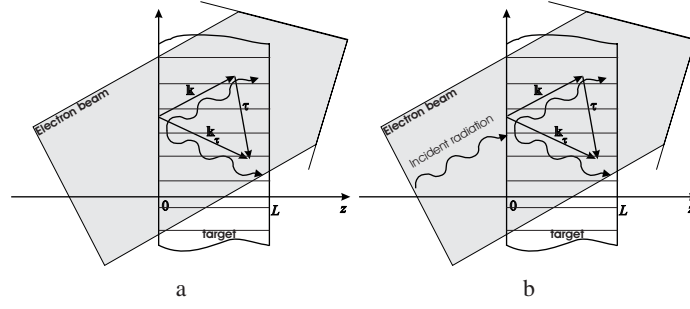


**Figure 3.** Electron beam current threshold in Bragg geometry.

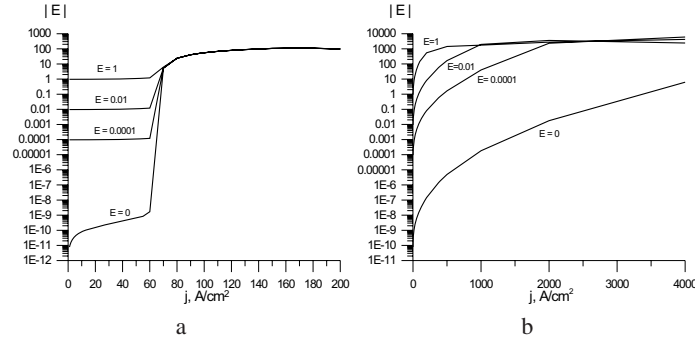
Two different geometries were studied: Bragg geometry (see Fig.1 and numerical results above), in which two waves propagate to opposite sides of resonator and Laue geometry, in which waves propagate to one side of a resonator (see Fig.4). Two regimes are possible in the Bragg case as stated above: regime of regenerative amplification and regime of generation. Two regimes are possible in Laue case too. There are amplification regime of emerging incident wave and regime SASE [15]. SASE (self amplified stimulated emission) develops from spontaneous noises (as well as generator regime in Bragg case). Since diffracted wave in Laue case propagates in the forward direction, we have to change right difference derivative in (3) for  $E_\tau$  to the left one. It is clear that boundary conditions for  $E_\tau$  in (1) should be written for  $z = 0$ .

Let us examine Fig. 5. We can see dependence between value of electric field and current density for different amplitudes of emerging waves for both diffraction geometries. Amplification regime corresponds to range of current density  $j$  from range  $60 \div 80$  A/cm<sup>2</sup> for Bragg geometry (Fig.5a). Current threshold is over-passed at  $j = 80$  A/cm<sup>2</sup>. This region corresponds to regime of generation. In Fig.5b the curve with  $E = 0$  corresponds to SASE regime. The rest curves demonstrate regime of amplification.

Let us consider two questions. First, where does radiation come from in the system in the absence of incident radiation? The answer is that it comes from spontaneous noises of electron beam. The second question asks what corresponds to this noise in numerical realization of mathematical model (1)–(3)? The answer is that such noise appears due to computational error on the right-hand side of equations. It



**Figure 4.** Scheme of quasi-Cherenkov VFEL in Laue geometry.



**Figure 5.** Dependence between computed electromagnetic wave amplitude and electron beam current density in Bragg (a) and Laue (b) geometry.

is clear that at the first iteration, when  $E(z) = 0$  all over  $z \in [0, L]$ , we have

$$I = \Phi \int_0^{2\pi} \frac{2\pi - p}{8\pi^2} \left( \exp(-i\theta(z, p)) + \exp(-i\theta(z, -p)) \right) dp \equiv 0.$$

But in fact we obtain  $I \sim 10^{-15}, 10^{-14}$ . This is an equivalent of spontaneous noises and fuse for the beginning of generation process.

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### Kvazi-Cherenkovo elektroninio spindulio nestabilumo modeliavimas periodinėse struktūrose

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Modeliuojama elektronų spindulio kvazi-Cherenkovo nestabilumo netiesinė fazė su dvimačio ir trimačio paskirstytojo grįžtamojo ryšio sąlyga. Nagrinėjama schema su grįžtamuoju ryšiu su dviem susietomis stipriomis bangomis. Pateiktas kvazi-Cherenkovo elektroninio spindulio nestabilumo matematinis modelis. Pasiūlytas veiksmingas skaitinis algoritmas, skirtas netiesinėms integro-diferencialinėms lygčių sistemoms su tokio tipo nestabilumu, spręsti. Apžvelgti skaitinio eksperimento rezultatai.